JOURNAL OF APPROXIMATION THEORY 70, 265–272 (1992)

On Characterization Theorems for Measures Associated with Orthogonal Systems of Rational Functions on the Unit Circle

K. PAN

Department of Mathematics, University of California Riverside, California 92521, U.S.A.

Communicated by Walter Van Assche

Received March 11, 1991; revised June 6, 1991

We establish a characterization theorem for log integrable measures associated with systems of rational functions that are orthonormal on the unit circle and have all their poles at a given sequence of points outside the unit disk. Ratio and nth root asymptotic theorems are proved as well. © 1992 Academic Press, Inc.

1. INTRODUCTION AND NOTATION

Let $d\mu$ be a finite positive Borel measure on the unit circle $\partial\Delta := \{z \in \mathbb{C}: |z| = 1\}$ whose support consists of infinitely many points. Let $\mu = \mu_a + \mu_s$ be its canonical decomposition into the absolutely continuous and the singular parts (with respect to Lebesgue measure on the circle). We denote by $\mu'(\theta)$ the Radon–Nikodym derivative of μ_a with respect to $d\theta$. Then $\mu' \in L_1[0, 2\pi)$ and $\mu'(\theta) \geqslant 0$ a.e. in $[0, 2\pi)$.

Let $\{\alpha_i\}_{i=0}^{\infty}$ with $|\alpha_i| < 1$ be an arbitrary sequence of complex numbers and let some of them have finite or even infinite multiplicity (it is not necessary for them to appear successively). The so-called Malmquist system of rational functions (cf. [19]) is defined by

$$\varphi_0(z) := \frac{(1 - |\alpha_0|^2)^{1/2}}{1 - \bar{\alpha}_0 z}$$

and

$$\varphi_n(z) := \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \bar{\alpha}_n z} \prod_{k=0}^{n-1} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k}, \qquad n = 1, ...,$$

0021-9045/92 $5.00

where for $\alpha_k = 0$ we put $|\alpha_k|/\alpha_k = -1$. It is easy to show that this Malmquist system is orthonormal on the unit circle in the sense that

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_m(z)\,\overline{\varphi_n(z)}\,d\theta = \delta_{m,n}, \qquad m, n = 0, 1, 2, ..., \; z = e^{i\theta}.$$

This system is the result of orthogonalization of the ordered sequence of functions of the system $\{\varphi_n(z)\}_0^\infty$ on the unit circle with the weight function $d\theta$.

Now denote by $\{\phi_n(z)\}_0^\infty$ the orthogonalization of the ordered systems of the Malmquist system on the unit circle with respect to $d\mu(\theta)$. Thus we come to the sequence of rational functions $\{\phi_n(z)\}_0^\infty$ which satisfies and is uniquely determined by the conditions

$$\phi_n(z) = \phi_n(d\mu, z) := \sum_{k=0}^{n} c_{k,n}\varphi_k(z), \qquad c_{n,n} =: \kappa_n > 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_m(z)\,\overline{\phi_n(z)}\,d\mu(\theta) = \delta_{m,n}, \qquad m, n = 0, 1, 2, ..., \; z = e^{i\theta}.$$

Note that in the extreme case, when $\alpha_k = 0$ $(0 \leqslant k < \infty)$, the system of functions $\{\phi_n(z)\}_0^\infty$ turns into the system of orthogonal polynomials on the unit circle with respect to $d\mu$. The reason for interest in this system is their close connection with the Nevanlinna–Pick interpolation problem and applications in circuits, signal processing, inverse scattering, systems theory, continued fractions, and Padé approximation. See for example [1–3, 5–10]. The most complete survey on the topic including the algebra as well as the analysis is given in an excellent report [4]. Here we prove some characterization theorems for the measure associated with $\{\phi_n(z)\}_0^\infty$ and the kernel function

$$S_n(\xi; z) = \sum_{k=0}^{n} \overline{\phi_k(\xi)}\,\phi_k(z), \qquad n = 0, 1, ...$$

We generalize the results in the extreme case (cf. [12, 14, 15]).

2. Main Theorems

We state our theorems here and give the proofs in the next section. We define

$$\hat{\varphi}_1(z) := \frac{(1 - |\alpha_1|^2)^{1/2}}{1 - \bar{\alpha}_1 z}$$

and

$$\hat{\varphi}_n(z) := \frac{(1-|\alpha_n|^2)^{1/2}}{1-\bar{\alpha}_n z} \prod_{k=1}^{n-1} \frac{\alpha_k - z}{1-\bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k}, \qquad n = 2, 3, \ldots .$$

For measures belonging to the Szegő class, we have the following characterization.

THEOREM 2.1. *Let* $\sum_{i=0}^{\infty} (1-|\alpha_i|) = \infty$. *Then the following statements are equivalent*:

(a) $\log \mu' \in L_1$.

(b) $\sum_{i=0}^{\infty} |\phi_i(0)|^2 < \infty$.

(c) *The set* $U := \{\hat{\varphi}_i\}_{i=1}^{\infty}$ *is not closed in* $L_2(d\mu)$.

Define $S_n(z) := S_n(0; z)$, $\gamma_n := S_n(0) = \sum_{i=0}^{n} |\phi_i(0)|^2$, and $w_n(z) := \prod_{i=0}^{n} (1-\bar{\alpha}_i z)$. Regarding the regularity of $d\mu$ (cf. [14, 17]), we have the following extension.

THEOREM 2.2. *For any measure* $d\mu$, $\lim_{n\to\infty} \gamma_n^{1/n} = 1$ *if and only if* $\lim_{n\to\infty} |w_n(z)\, S_n(z)|^{1/n} = 1$ *locally uniformly in* $|z| < 1$.

Remark. Note that for the case $\alpha_k = 0$, $k = 0, 1, \ldots$, $S_n(z) = \kappa_n \phi_n^*(z)$ and $\gamma_n = \kappa_n^2$. So Theorem 3.3 in [14] is a special case of Theorem 2.2.

For the ratio asymptotic behavior, we have

THEOREM 2.3. *For any measure* $d\mu$, $\lim_{n\to\infty} \gamma_{n-1}/\gamma_n = 1$ *if and only if* $\lim_{n\to\infty} S_{n-1}(z)/S_n(z) = 1$ *locally uniformly in* $|z| < 1$.

3. PROOF OF THEOREMS

Let $\mathscr{P}_n$ denote the set of all polynomials with degree at most n and let $\mathscr{P}_n[\alpha_0, \ldots, \alpha_n]$ be the set of all the linear combinations of the first $n+1$ Malmquist functions $\varphi_0(z), \ldots, \varphi_n(z)$. It is easy to see that any $u_n \in \mathscr{P}_n[\alpha_0, \ldots, \alpha_n]$ can be written as $u_n(z) = v_n(z)/w_n(z)$, where $v_n \in \mathscr{P}_n$. For any $p_n(z) = a_n z^n + \cdots$, $a_n \neq 0$, define $p_n^*(z) := z^n \overline{p_n(1/\bar{z})}$.

Proof of Theorem 2.1. (a) $\Rightarrow$ (b): The proof follows from Theorem 3.4 in [10].

(b) $\Rightarrow$ (c): Let

$$\mathscr{A} := \{R_n(z) \colon R_n(z) \in \mathscr{P}_n[\alpha_0, \ldots, \alpha_n] \text{ and } R_n(0) = 1\}$$

and

$$\chi(R_n) := \frac{1}{2\pi} \int_0^{2\pi} |R_n(z)|^2 \, d\mu(\theta), \qquad z = e^{i\theta}.$$

Then from [7], we know that

$$\chi(S_n/\gamma_n) = \min_{R_n \in \mathscr{A}} \chi(R_n) = 1/\gamma_n. \tag{3.1}$$

For any $R_n \in \mathscr{A}$, since $R_n(0) = 1$, we can rewrite $R_n(z)$ as $r_n(z)/w_n(z)$, where $r_n \in \mathscr{P}_n$ and $r_n(0) = 1$, and so $R_n(z) = [1 + zp_{n-1}(z)]/w_n(z)$ for some $p_{n-1} \in \mathscr{P}_{n-1}$. From (3.1) we can see that

$$\begin{aligned} 1/\gamma_n &= \min_{R_n \in \mathscr{A}} \chi(R_n) = \min_{p_{n-1} \in \mathscr{P}_{n-1}} \chi([1 + zp_{n-1}(z)]/w_n(z)) \\ &= \min_{p_{n-1} \in \mathscr{P}_{n-1}} \chi(\{1 + [1 + zp_{n-1}(z) - \hat{w}_n(z)]/\hat{w}_n(z)\}/(1 - \bar{\alpha}_0 z)), \end{aligned}$$

where $\hat{w}_n(z) := w_n(z)/(1 - \bar{\alpha}_0 z)$. Note that $q_n(z) := 1 + zp_{n-1}(z) - \hat{w}_n(z) \in \mathscr{P}_n$ and $q_n(0) = 0$, so $q_n(z) = zt_{n-1}(z)$ for some $t_{n-1} \in \mathscr{P}_{n-1}$. Thus

$$\begin{aligned} 1/\gamma_n &\leqslant \min_{t_{n-1} \in \mathscr{P}_{n-1}} \chi(1 + [zt_{n-1}(z)]/\hat{w}_n(z))/(1 - |\alpha_0|)^2 \\ &= \min_{t_{n-1} \in \mathscr{P}_{n-1}} \chi(1/z + t_{n-1}(z)/\hat{w}_n(z))/(1 - |\alpha_0|)^2 \\ &= \min_{u_{n-1} \in \mathscr{P}_{n-1}[\alpha_1, ..., \alpha_n]} \chi(1/z + u_{n-1}(z))/(1 - |\alpha_0|)^2. \end{aligned}$$

Thus from (b) we know that the function $1/z$ cannot be approximated by U to arbitrary accuracy in $L_2(d\mu)$.

(c) $\Rightarrow$ (a): Let us assume that U is not closed; then there is a function $\xi(\theta) \in L_2(d\mu)$, $\xi \not\equiv 0$ and orthogonal to all functions of U, i.e., satisfying the conditions

$$\frac{1}{2\pi} \int_0^{2\pi} \hat{\varphi}_k(z) \, \overline{\xi(\theta)} \, d\mu(\theta) = 0, \qquad k = 1, 2, ..., \; z = e^{i\theta}. \tag{3.2}$$

For any $|\alpha| > 1$, $(z - \alpha)^{-1}$ is analytic in $|z| \leqslant 1$. So from $\sum_{i=1}^{\infty} (1 - |\alpha_i|) = \infty$ and the theorem in [19, p. 306], we have

$$(z - \alpha)^{-1} = \sum_{i=1}^{\infty} c_i \hat{\varphi}_i(z)$$

uniformly on $|z| = 1$ for some c_i. From (3.2), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{\xi(\theta)} \, d\mu(\theta)}{z - \alpha} = 0, \qquad z = e^{i\theta}, \; |\alpha| > 1.$$

We introduce the notation

$$\lambda(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\frac{z\,d\tau(\theta)}{z-\alpha}, \qquad d\tau(\theta)=z^{-1}\overline{\xi(\theta)}\,d\mu(\theta),\ z=e^{i\theta}.$$

This function has the same properties as the function $\lambda(z)$ in [12, p. 15]. Using the same argument in [12, p. 15] we know that $\log\mu'(\theta)\in L_1$. ∎

Proof of Theorem 2.2. Let us rewrite $S_n(z)=s_n(z)/w_n(z)$, where $s_n\in\mathscr{P}_n$ and $s_n(0)=\gamma_n$. From (3.1) we can see that all the zeros of s_n are outside of $|z|\leqslant 1$. We now consider the new function $s_n^*(z)=\gamma_n z^n+\cdots+a_n$; all its zeros lie in $|z|<1$. By Helly's Selection Theorem and an ordinary compactness argument we know that any infinite subsequence $\Lambda\subseteq\mathbf{N}$ contains an infinite subsequence, which we continue to denote by Λ, so that the two limits

$$\frac{1}{n}\nu_{s_n^*}\xrightarrow{*}\nu \text{ and } \frac{1}{n}\log\gamma_n\to 0, \qquad n\to\infty,\ n\in\Lambda,$$

exist for some measure ν with compact $\operatorname{supp}(\nu)\subset\mathbf{C}$, where $\nu_{s_n^*}$ is a discrete measure with mass one at each zero of s_n^*. From the proof of Theorem 3.1.1 in [17], we have

$$\lim_{n\to\infty}|s_n^*(z)|^{1/n}=|z|,$$

locally uniformly in $|z|>1$. That is,

$$\lim_{n\to\infty}|s_n(z)|^{1/n}=1,$$

and so

$$\lim_{n\to\infty}|w_n(z)\,S_n(z)|^{1/n}=1,$$

locally uniformly in $|z|<1$. The proof of the other direction is trivial. ∎

Proof of Theorem 2.3. From (3.1), we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&\left|\frac{s_n^*(z)}{\gamma_n}\right|^2\frac{d\mu(\theta)}{|w_n(z)|^2}\\&=\frac{1}{2\pi}\int_0^{2\pi}|S_n(z)/\gamma_n|^2\,d\mu(\theta)\\&=\min_{R_n\in\mathscr{A}}\chi(R_n)=\min_{\substack{p_n\in\mathscr{P}_n\\ p_n(0)=1}}\frac{1}{2\pi}\int_0^{2\pi}|p_n(z)|^2\frac{d\mu(\theta)}{|w_n(z)|^2}\\&=\min_{p_{n-1}\in\mathscr{P}_{n-1}}\frac{1}{2\pi}\int_0^{2\pi}|z^n+p_{n-1}(z)|^2\frac{d\mu(\theta)}{|w_n(z)|^2}, \qquad z=e^{i\theta},\end{aligned}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{s_n^*(z)}{\sqrt{\gamma_n}}\right|^2 \frac{d\mu(\theta)}{|w_n(z)|^2}=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{S_n(z)}{\sqrt{\gamma_n}}\right|^2 d\mu(\theta)=1.$$

Thus $s_n^*(z)/\sqrt{\gamma_n}$ is the nth orthonormal polynomial with respect to $d\mu/|w_n(z)|^2$. From the theorem in [11, p. 198], we have

$$\begin{aligned}\frac{1}{2\pi}&\int_0^{2\pi}\left|\frac{s_{n-1}(z)(1-\bar{\alpha}_n z)}{s_n(z)/\sqrt{\gamma_n}}\right|^2 d\theta\\ &=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{s_{n-1}(z)\,(1-\bar{\alpha}_n z)}{s_n^*(z)/\sqrt{\gamma_n}}\right|^2 d\theta\\ &=\frac{1}{2\pi}\int_0^{2\pi}|s_{n-1}(z)(1-\bar{\alpha}_n z)|^2\frac{d\mu(\theta)}{|w_n(z)|^2}\\ &=\frac{1}{2\pi}\int_0^{2\pi}|S_{n-1}(z)|^2\,d\mu(\theta)=\gamma_{n-1}.\end{aligned}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{s_{n-1}(z)(1-\bar{\alpha}_n z)}{s_n(z)}\right|^2 d\theta=\gamma_{n-1}/\gamma_n. \tag{3.3}$$

Again all the zeros of s_n lie in $|z|>1$, so the function $s_{n-1}(z)(1-\bar{\alpha}_n z)/s_n(z)$ is analytic in $|z|\leqslant 1$ and has the following expansion

$$\frac{s_{n-1}(z)(1-\bar{\alpha}_n z)}{s_n(z)}=\gamma_{n-1}/\gamma_n+a_{n,1}z+a_{n,2}z^2+\cdots.$$

From (3.3), we can conclude that

$$|\gamma_{n-1}/\gamma_n|^2+|a_{n,1}|^2+|a_{n,2}|^2+\cdots=\gamma_{n-1}/\gamma_n.$$

In consequence of Cauchy's inequality this yields, for $|z|<1$,

$$\begin{aligned}|a_{n,1}z+a_{n,2}z^2+\cdots|^2&\leqslant(|a_{n,1}|^2+|a_{n,2}|^2+\cdots)\frac{|z|^2}{1-|z|^2}\\ &\leqslant(1-\gamma_{n-1}/\gamma_n)\frac{|z|^2}{1-|z|^2}\end{aligned}$$

and as $n\to\infty$, together with $\lim_{n\to\infty}\gamma_{n-1}/\gamma_n=1$, the last expression tends to 0 uniformly in $|z| \leqslant r < 1$, and so we have $\lim_{n \to \infty} s_{n-1}(z)(1 - \bar{\alpha}_n z)/s_n(z) = 1$, i.e., $\lim_{n \to \infty} S_{n-1}(z)/S_n(z) = 1$ locally uniformly in $|z| < 1$. The proof of the other direction is easy. ∎

ACKNOWLEDGMENTS

We thank A. Bultheel, E. B. Saff, W. Van Assche, and the referee for bringing some references to our attention and for their helpful suggestions.

REFERENCES

1. A. BULTHEEL, Orthogonal matrix functions related to the multivariable Nevanlinna–Pick problem, *Bull. Soc. Math. Belg. Sér. B* **32** (1980), 149–170.
2. A. BULTHEEL, On a special Laurent–Hermite interpolation problem, *in* "Numerical Methods of Approximation Theory" (L. Colatz, G. Meinardus, and H. Werner, Eds.), pp. 63–79, International Series in Numerical Mathematics, No. 6, Birkhäuser, Basel, 1981.
3. A. BULTHEEL AND P. DEWILDE, Orthogonal functions related to the Nevanlinna–Pick problem, *in* "Mathematical Theory of Networks and Systems" (P. Dewilde, Ed.), pp. 207–211, Proceedings, Mathematical Theory of Networks and Systems Conference, Delft, 1979, Western Periodicals, North Hollywood, CA, 1979.
4. A. BULTHEEL, P. GONZÁLEZ-VERA, E. HENDRIKSEN, AND O. NJÅSTAD, "A Szegő Theory for Rational Functions," Report TW 131, Department of Computer Science, Katholieke Universiteit, Leuven, May 1990.
5. A. BULTHEEL, P. GONZÁLEZ-VERA, E. HENDRIKSEN, AND O. NJÅSTAD, Orthogonal functions similar to Szegő polynomials, *in* "IMACS Transactions on Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, *et al.*, Eds.), Birkhäuser, Basel, 1990.
6. P. DEWILDE, A. VIEIRA, AND T. KAILATH, On the generalized Szegő-Levinson realization algorithm for optimal linear predictors based on network synthesis approach, *IEEE Trans. Circuits Systems* **25** (1978), 663–675.
7. M. M. DJRBASHIAN, Orthogonal systems of rational functions on the cirlce with given set of poles, *Dokl. SSSR* **147** (1961), 1278–1281.
8. M. M. DJRBASHIAN, Orthogonal systems of rational functions on the circle, *Izv. Akad. Nauk. Armyan. SSR Ser. Mat.* **1** (1966), 3–24, **1** (1966), 106–125.
9. M. M. DJRBASHIAN, Representation and completeness of some orthogonal systems, *Izv. Akad. Nauk. Armyan. SSR Ser. Mat.* **4** (1979), 445–493.
10. M. M. DJRBASHIAN, A survey on the theory of orthogonal systems and some open problems, *in* "Orthogonal Polynomials: Theory and Practice" (Paul Nevai, Ed.), pp. 135–146, NATO-ASI Series.
11. G. FREUD, "Orthogonal Polynomials," Pergamon, New York, 1971.
12. L. YA. GERONIMUS, "Orthogonal Polynomials," Consultants Bureau, New York, 1961.
13. E. HENDRIKSEN AND O. NJÅSTAD, Positive multipoint Padé continued fractions, *Proc. Edinburgh Math. Soc.* **32** (1989), 261–269.
14. X. LI, E. B. SAFF, AND Z. SHA, Behavior of best L_p polynomial approximants on the unit interval and on the unit circle, *J. Approx. Theory* **63** (1990), 170–190.
15. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Asymptotics for the ratio of leading coefficients of orthonormal polynomials on the unit circle, *Constr. Approx.* **1** (1985), 63–69.
16. O. NJÅSTAD, Multipoint Padé approximation and orthogonal rational functions, *in*

"Nonlinear Numerical Methods and Rational Approximation" (A. Cuyt, Ed.), pp. 258–270, Reidel, Dordrecht, 1988.

17. H. STAHL AND V. TOTIK, "General Orthogonal Polynomials," manuscript.
18. G. SZEGŐ, "Orthogonal Polynomials," 4th ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc. Providence, RI, 1975.
19. J. L. WALSH, "Interpolation and Approximation by Rational Functions in the Complex Domain," 5th ed., Americal Mathematical Colloquium Publications, Vol. 20, Amer. Math. Soc., Providence, RI, 1969.